

# $k$ -Geometric Mean Graphs

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## Abstract

A finite, simple and undirected graph  $G = (V, E)$  with  $p$  vertices and  $q$  edges is said to be a *k-geometric mean graph* for a positive integer  $k$  if there is an injection  $\psi : V(G) \rightarrow \{k, k+1, \dots, k+q\}$  such that, when each edge  $uv \in E(G)$  is assigned the label  $\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$  or  $\lceil \sqrt{\psi(u)\psi(v)} \rceil$ , the resulting edge label set is  $\{k, k+1, \dots, k+q-1\}$  and  $\psi$  is called a *k-geometric mean labeling* of  $G$ . The special case  $k = 1$ , a 1-geometric mean labeling is called a geometric mean labeling, and a 1-geometric mean graph is called a geometric mean graph.

In this paper, we present new classes of geometric mean graphs. Then we introduce *k-geometric mean labeling* and prove some classes of graphs are *k-geometric mean*. We also study some classes of finite join of graphs that admit geometric mean labeling.

## 1 Introduction

In this paper, we let  $G$  be a finite, simple and undirected graph with vertex set  $V(G)$  and edge set  $E(G)$ , where  $|V(G)| = p$  and  $|E(G)| = q$ .

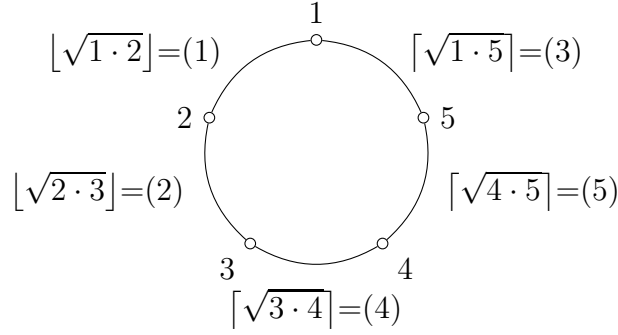
Geometric mean labeling was first introduced by Somasundaram et al. (2011) in [2]. Many classes of geometric mean graphs are studied in [2], [3], [4], [5], [6], and [7]. Durai Baskar and Arockiaraj give a nice motivation and introduce the concept of  $\mathcal{F}$ -geometric mean labeling, where only the flooring function is used, in [1]. We here investigate further into new classes of geometric mean graphs and a finite join of them.

**Definition 1.** Let  $G = (V, E)$  be a graph with  $p$  vertices and  $q$  edges.

The graph  $G$  is said to be a *geometric mean graph* if there is an injection  $\psi : V(G) \rightarrow \{1, 2, \dots, q+1\}$  such that, when each edge  $uv \in E(G)$  is assigned the label  $\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$  or  $\lceil \sqrt{\psi(u)\psi(v)} \rceil$ , the resulting edge label set is  $\{1, 2, \dots, q\}$  and  $\psi$  is called a *geometric mean labeling* of  $G$ .

Here  $P_n$ ,  $C_n$ ,  $K_n$ , and  $S_n$  are the path on  $n$  vertices, cycle on  $n$  vertices, complete graph on  $n$  vertices, and star graph on  $n + 1$  vertices, respectively.

**Example 1.** A geometric mean labeling of  $C_5$  is given below.



**Definition 2.** The *union*  $G_1 \cup G_2$  of two graphs  $G_1$  and  $G_2$  is the graph  $G$  with  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2)$ .

**Definition 3.** The *join*  $G_1 + G_2$  of two graphs  $G_1$  and  $G_2$  with disjoint set of vertices is the union of two graphs  $G_1$  and  $G_2$ .

**Definition 4.** The *corona*  $G_1 \odot G_2$  of two graphs  $G_1$  with  $p$  vertices and  $G_2$  is the graph  $G$  obtained by taking one copy of  $G_1$  and  $p$  copies of  $G_2$  and then joining the  $i^{th}$  vertex of  $G_1$  to every vertices in the  $i^{th}$  copy of  $G_2$ .

**Theorem 1** ([2]). *Any path is a geometric mean graph.*

**Theorem 2** ([2]). *Any cycle is a geometric mean graph.*

**Definition 5.** A *comb* is the graph  $P_n \odot K_1$ .

**Theorem 3** ([2]). *Any comb is a geometric mean graph.*

**Definition 6.** A *crown* is the graph  $C_n \odot K_1$ .

**Theorem 4** ([3]). *Any crown is a geometric mean graph.*

**Definition 7.** A *triangular snake*  $T_n$  is obtained from replacing each edge in  $P_n$  with  $C_3$ .

**Theorem 5** ([6]). *Any triangular snake  $T_n$  is a geometric mean graph.*

**Definition 8.** A *quadrilateral snake*  $Q_n$  is obtained from replacing each edge in  $P_n$  with  $C_4$ .

**Theorem 6** ([6]). *Any quadrilateral snake  $Q_n$  is a geometric mean graph.*

Somasundram et al. provide some results on the join of graphs in [2], [3] and [6] as follows.

**Theorem 7** ([3]).  $C_m + P_n$  is a geometric mean graph.

**Theorem 8** ([3]).  $C_m + C_n$  is a geometric mean graph.

**Theorem 9** ([3]). For any  $n \in \mathbb{N}$ ,  $n$  disjoint copy of  $C_3$  denoted by  $nC_3$  is a geometric mean graph.

**Theorem 10** ([3]). For any  $m, n \in \mathbb{N}$ ,  $nC_3 + P_m$  is a geometric mean graph.

**Theorem 11** ([3]). For any  $m, n \in \mathbb{N}$ ,  $nC_3 + C_m$  is a geometric mean graph.

**Theorem 12** ([6]).  $C_m + T_n$  is a geometric mean graph.

**Theorem 13** ([6]).  $(C_m \odot K_1) + T_n$  is a geometric mean graph.

**Theorem 14** ([6]).  $C_m + Q_n$  is a geometric mean graph.

**Theorem 15** ([6]).  $(C_m \odot K_1) + Q_n$  is a geometric mean graph.

Here we give some new results on geometric mean graphs and establish a general result on finite join of paths, cycles, combs, crowns, triangle snakes, and quadrilateral snakes.

## 2 Main results

### 2.1 Stars graphs

**Lemma 16.** Let  $G$  be a graph. If  $|V(G)| > |E(G)| + 1$ , then  $G$  is not a geometric mean graph.

*Proof.* If  $|V(G)| > |E(G)| + 1$ , then the injective vertex labeling does not exist and hence  $G$  cannot be a geometric mean graph.  $\square$

**Theorem 17.** The join of any two graphs from the set of trees (including paths, combs, and stars), triangle snakes, and quadrilateral snakes, is not a geometric mean graph.

*Proof.* Let  $G$  be a join of graphs  $G_1$  and  $G_2$ , which are from the set of trees, triangle snakes, and quadrilateral snakes. Then

$$\begin{aligned} |V(G)| &= |V(G)_1| + |V(G_2)| \\ &\geq (|E(G)_1| + 1) + (|V(G_2)| + 1) \\ &> |E(G)_1| + |E(G_2)| + 1 \\ &= |E(G)| + 1. \end{aligned}$$

By Lemma 16,  $G$  is not a geometric mean graph.  $\square$

**Theorem 18.** *If the star  $S_n$  is a geometric mean graph then  $n \leq 7$ .*

*Proof.* A star  $S_n$  has exactly  $n + 1$  vertices and  $n$  edges. Hence all labels in  $\{1, 2, \dots, n + 1\}$  are used on vertices labeling.

Assume that there exists a geometric mean labeling of  $S_n$ . Let the only vertex of degree greater than 1 be labelled by  $k \in \{1, 2, \dots, n + 1\}$ . Therefore the lowest possible edge label is  $\lfloor \sqrt{1 \cdot k} \rfloor$  and the highest possible edge label is  $\lceil \sqrt{k \cdot (n + 1)} \rceil$

Since the graph need at least  $n$  edge labels. It is nessesary that

$$\lceil \sqrt{k \cdot (n + 1)} \rceil - \lfloor \sqrt{1 \cdot k} \rfloor + 1 \geq n. \quad (\star)$$

Since  $\lfloor \sqrt{1 \cdot k} \rfloor \geq 1$ , to achieve  $(\star)$ , we need  $\lceil \sqrt{k \cdot (n + 1)} \rceil$  to be greater than  $n$ . Thus  $k$  is required to be in  $\{n - 2, n - 1, n, n + 1\}$ .

**Case  $k = n - 2$  or  $n - 1$ ;**

Consider

$$\begin{aligned} n &\leq \lceil \sqrt{k \cdot (n + 1)} \rceil - \lfloor \sqrt{1 \cdot k} \rfloor + 1 \\ &\leq n - \lfloor \sqrt{1 \cdot (n - 2)} \rfloor + 1. \end{aligned}$$

Hence  $\lfloor \sqrt{1 \cdot (n - 2)} \rfloor = 1$ . Which implies  $n \in \{1, 2, \dots, 5\}$ .

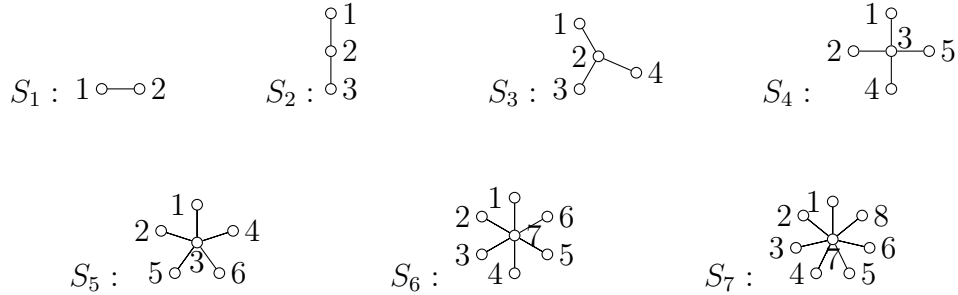
**Case  $k = n$  or  $n + 1$ ;**

Consider

$$\begin{aligned} n &\leq \lceil \sqrt{k \cdot (n + 1)} \rceil - \lfloor \sqrt{1 \cdot k} \rfloor + 1 \\ &\leq (n + 1) - \lfloor \sqrt{1 \cdot n} \rfloor + 1. \end{aligned}$$

Hence  $\lfloor \sqrt{1 \cdot n} \rfloor = 2$  or  $1$ . Which implies  $n \in \{1, 2, \dots, 8\}$ . However, without too much effort, one can see that when  $n = 8$ , both  $k = 8$  and  $k = 9$  do not give a geometric mean labeling.  $\square$

**Example 2.** Here are examples of geometric mean labeling of  $S_n$  for  $n = 1$  to 7.



## 2.2 $k$ -geometric mean graphs

Here we define  $k$ -geometric mean labeling as a tool to proof our main theorem.

**Definition 9.** A graph  $G = (V, E)$  with  $p$  vertices and  $q$  edges is said to be a  $k$ -geometric mean graph ( $k$  is a positive integer) if there is an injection  $\psi : V(G) \rightarrow \{k, k+1, \dots, k+q\}$  such that, when each edge  $uv \in E(G)$  is assigned the label  $\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$  or  $\lceil \sqrt{\psi(u)\psi(v)} \rceil$ , the resulting edge label set is  $\{k, k+1, \dots, k+q-1\}$  and  $\psi$  is called a  $k$ -geometric mean labeling of  $G$ .

Note that from Definition 9, a geometric mean labeling is a 1-geometric mean labeling.

**Theorem 19.** Let  $n, k \in \mathbb{N}$ ,  $P_n$  is a  $k$ -geometric mean graph.

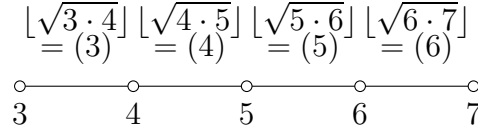
*Proof.* Let  $P_n$  be the path  $u_1 u_2 \dots u_n$ .

Define a function  $\psi : V(P_n) \rightarrow \{k, k+1, \dots, k+(n-1) = k+q\}$  by

$$\psi(u_i) = (k-1) + i, \quad 1 \leq i \leq n$$

For  $i = 1$  to  $n-1$ , assign the label  $\lfloor \sqrt{\psi(u_i)\psi(u_{i+1})} \rfloor$  to edge  $u_i u_{i+1}$ . Then the set of edge labels is  $\{k, k+1, \dots, k+(n-1)-1 = k+q-1\}$ . Hence  $P_n$  is a  $k$ -geometric mean graph.  $\square$

**Example 3.** A 3-geometric mean labeling of  $P_5$  is given below.



**Theorem 20.** Let  $n, k \in \mathbb{N}$  and  $n \geq 3$ ,  $C_n$  is a  $k$ -geometric mean graph.

*Proof.* Let  $C_n$  be the cycle  $u_1 u_2 \dots u_n u_1$ .

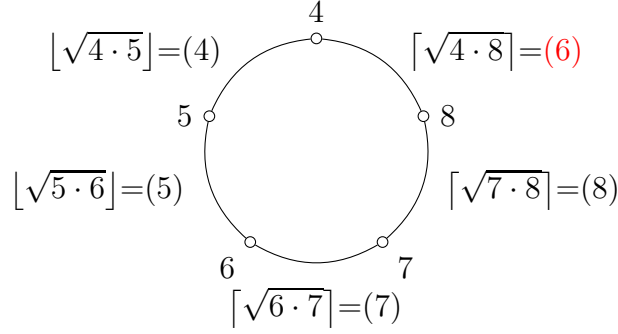
Define a function  $\psi : V(C_n) \rightarrow \{k, k+1, \dots, k+n = k+q\}$  by

$$\psi(u_i) = (k-1) + i, \quad 1 \leq i \leq n$$

Let  $h = \lceil \sqrt{k(k+n)} \rceil$ . We have  $k < h < k+n-1$ .

Assign the label  $h$  to edge  $u_1 u_n$ . For each edge  $u_i u_{i+1}$  assign the label  $\lfloor \sqrt{\psi(u_i)\psi(u_{i+1})} \rfloor$  when  $1 \leq i \leq h-k-1$ , assign the label  $\lceil \sqrt{\psi(u_i)\psi(u_{i+1})} \rceil$  otherwise. Then the set of edge labels is  $\{k, k+1, \dots, k+n-1 = k+q-1\}$ . Hence  $C_n$  is a  $k$ -geometric mean graph.  $\square$

**Example 4.** A 4-geometric mean labeling of  $C_5$  is given below.



**Theorem 21.** Let  $n, k \in \mathbb{N}$  and  $n \geq 3$ ,  $C_n \odot K_1$  is a  $k$ -geometric mean graph.

*Proof.* Let  $C_n$  be the cycle  $u_1 u_2 \dots u_n u_1$  and let  $v_i$  be the vertex adjacent to  $u_i$  for  $1 \leq i \leq n$ .

Define a function  $\psi : V(C_n \odot K_1) \rightarrow \{k, k+1, \dots, k+2n = k+q\}$  by

$$\begin{aligned} \psi(u_i) &= (k-1) + 2i, & 1 \leq i \leq n \\ \psi(v_i) &= (k-1) + 2i - 1, & 1 \leq i \leq n \end{aligned}$$

Let  $h = \left\lceil \sqrt{(k+1)(k+2n-1)} \right\rceil$ . We have  $k+1 < h < k+2n-1$ .

Assign the label  $h$  to edge  $u_1 u_n$ . For each edge  $uv$  in  $C_n \odot K_1$  assign the label as follows

**Case 1:**  $h = (k-1) + 2j = k+2j-1$  for some  $j \in \{2, 3, \dots, n-1\}$

edge $uv$	label	obtained labels
$u_i u_{i+1}, \quad 1 \leq i \leq j-1$	$\left\lfloor \sqrt{\psi(u)\psi(v)} \right\rfloor$	$\{k+1, k+3, \dots, k+2j-3\}$
$u_i u_{i+1}, \quad j+1 \leq i \leq n-1$	$\left\lceil \sqrt{\psi(u)\psi(v)} \right\rceil$	$\{k+2j, k+2j+2, \dots, k+2n-2\}$
$u_i v_i, \quad 1 \leq i \leq j$	$\left\lfloor \sqrt{\psi(u)\psi(v)} \right\rfloor$	$\{k, k+2, \dots, k+2j-2\}$
$u_i v_i, \quad j+1 \leq i \leq n$	$\left\lceil \sqrt{\psi(u)\psi(v)} \right\rceil$	$\{k+2j+1, k+2j+3, \dots, k+2n-1\}$

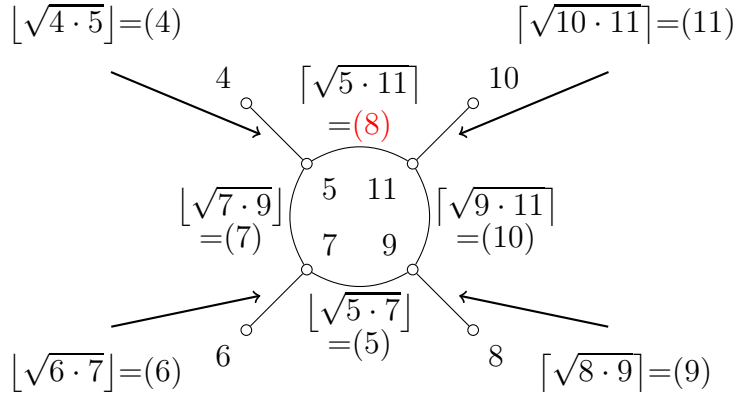
**Case 2:**  $h = (k-1) + 2j - 1 = k+2j-2$  for some  $j \in \{2, 3, \dots, n-1\}$

edge $uv$	label	obtained labels
$u_i u_{i+1}, \quad 1 \leq i \leq j-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k+1, k+3, \dots, k+2j-3\}$
$u_i u_{i+1}, \quad j+1 \leq i \leq n-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k+2j, k+2j+2, \dots, k+2n-2\}$
$u_i v_i, \quad 1 \leq i \leq j-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k, k+2, \dots, k+2j-4\}$
$u_i v_i, \quad j \leq i \leq n$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k+2j-1, k+2j+1, \dots, k+2n-1\}$

The sets of edge labels in both cases are  $\{k, k+1, \dots, k+2n-1 = k+q-1\}$ . Hence  $C_n \odot K_1$  is a  $k$ -geometric mean graph.  $\square$

Observe that the number of vertices and the number of edges are equal for cycles and crowns. Moreover, we can label vertices of cycles and crown without using the label  $k+q$ .

**Example 5.** A 4-geometric mean labeling of  $C_4 \odot K_1$  is given below.



**Theorem 22.** For any  $n \in \mathbb{N}$ ,  $P_n \odot K_1$  is a  $k$ -geometric mean graph.

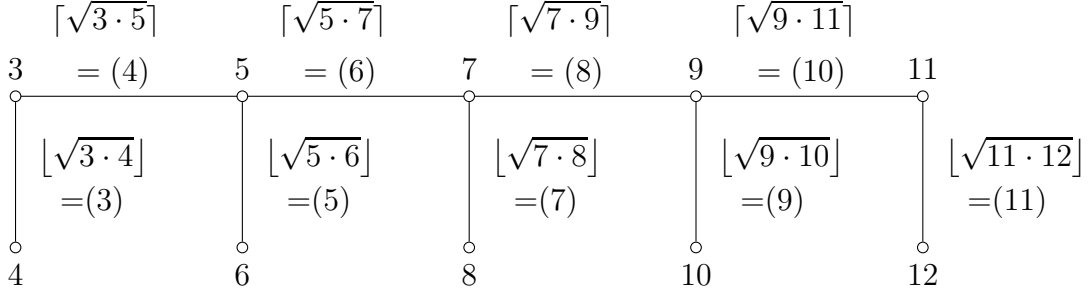
*Proof.* Let  $P_n$  be the path  $u_1 u_2 \dots u_n$  and let  $v_i$  be the vertex adjacent to  $u_i$  for  $1 \leq i \leq n$ .

Define a function  $\psi : V(P_n \odot K_1) \rightarrow \{k, k+1, \dots, k+2n-1 = k+q\}$  by

$$\begin{aligned} \psi(u_i) &= (k-1) + 2i - 1, & 1 \leq i \leq n \\ \psi(v_i) &= (k-1) + 2i, & 1 \leq i \leq n \end{aligned}$$

Assign the label  $\lfloor \sqrt{\psi(u_i)\psi(u_{i+1})} \rfloor$  to edge  $u_i u_{i+1}$ , for  $1 \leq i \leq n-1$ , and assign the label  $\lfloor \sqrt{\psi(u_i)\psi(v_i)} \rfloor$  to edge  $u_i v_i$ , for  $1 \leq i \leq n$ . Then the set of edge labels is  $\{k, k+1, \dots, k+2n-2 = k+q-1\}$ . Hence  $P_n \odot K_1$  is a  $k$ -geometric mean graph.  $\square$

**Example 6.** A 3-geometric mean labeling of  $P_5 \odot K_1$  is given below.



**Theorem 23.** For any  $n \in \mathbb{N}$ ,  $T_n$  is a  $k$ -geometric mean graph.

*Proof.* Let  $P_n$  be the path  $u_1 u_2 \dots u_n$  and let  $T_n$  be the triangular snake obtained from the path  $P_n$  by joining  $u_i$  and  $u_{i+1}$  to new vertex  $v_i$ , for  $1 \leq i \leq n-1$ .

Define a function  $\psi : V(T_n) \rightarrow \{k, k+1, \dots, k+3n-3 = k+q\}$  by

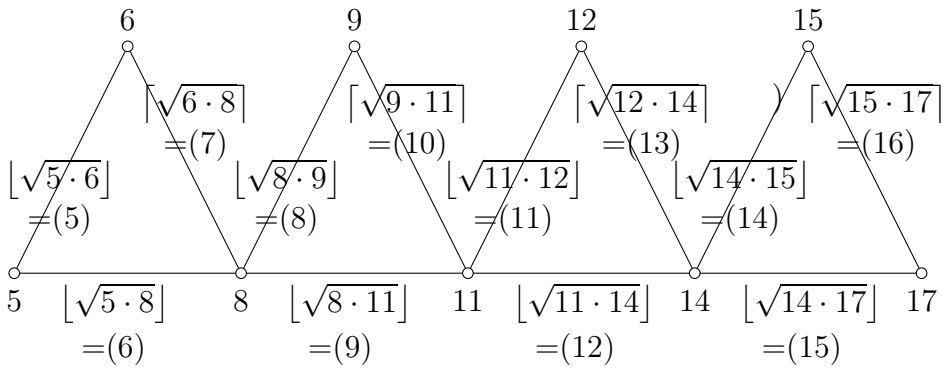
$$\begin{aligned} \psi(u_i) &= (k-1) + 3i - 2, & 1 \leq i \leq n \\ \psi(v_i) &= (k-1) + 3i - 1, & 1 \leq i \leq n-1 \end{aligned}$$

For each edge in  $T_n$  assign the label as follows

edge $uv$	label	obtained labels
$u_i u_{i+1}, \quad 1 \leq i \leq n-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k+1, k+4, \dots, k+3n-5\}$
$u_i v_i, \quad 1 \leq i \leq n-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k, k+3, k+6, \dots, k+3n-6\}$
$v_i u_{i+1}, \quad 1 \leq i \leq n-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k+2, k+5, \dots, k+3n-4\}$

Then the set of edge labels is  $\{k, k+1, \dots, k+3n-4 = k+q-1\}$ . Hence  $T_n$  is a  $k$ -geometric mean graph.  $\square$

**Example 7.** A 5-geometric mean labeling of  $T_5$  is given below.





**Theorem 24.** For any  $n \in \mathbb{N}$ ,  $Q_n$  is a  $k$ -geometric mean graph.

*Proof.* Let  $P_n$  be the path  $u_1 u_2 \dots u_n$  and let  $Q_n$  be the quadrilateral snake obtained from the path  $P_n$  by joining  $u_i$  and  $u_{i+1}$  to new vertices  $v_i, w_i$  in such a way that  $u_i v_i w_i v_{i+1}$  is a path, for  $1 \leq i \leq n-1$ .

Define a function  $\psi : V(Q_n) \rightarrow \{k, k+1, \dots, k+4n-4 = k+q\}$  by

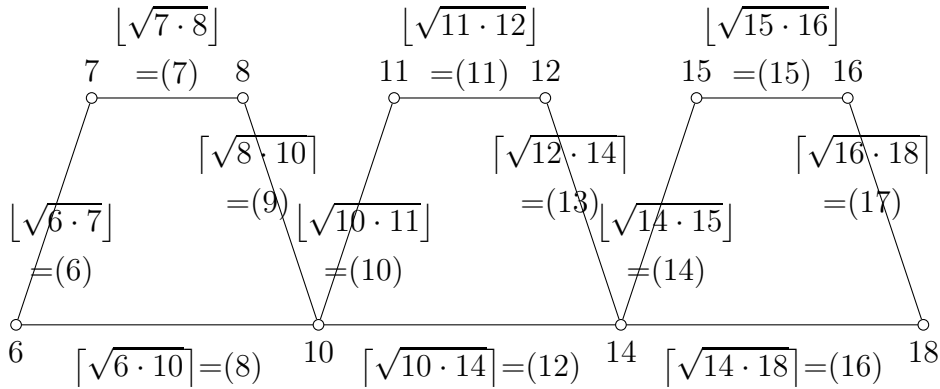
$$\begin{aligned}\psi(u_i) &= (k-1) + 4i - 3, & 1 \leq i \leq n \\ \psi(v_i) &= (k-1) + 4i - 2, & 1 \leq i \leq n-1 \\ \psi(w_i) &= (k-1) + 4i - 1, & 1 \leq i \leq n-1\end{aligned}$$

For each edge in  $Q_n$  assign the label as follows

edge $uv$	label	obtained labels
$u_i u_{i+1}, \quad 1 \leq i \leq n-1$	$\lceil \sqrt{\psi(u)\psi(v)} \rceil$	$\{k+2, k+6, \dots, k+4n-6\}$
$u_i v_i, \quad 1 \leq i \leq n-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k, k+4, k+8, \dots, k+4n-8\}$
$v_i w_i, \quad 1 \leq i \leq n-1$	$\lfloor \sqrt{\psi(u)\psi(v)} \rfloor$	$\{k+1, k+5, \dots, k+4n-7\}$
$w_i u_{i+1}, \quad 1 \leq i \leq n-1$	$\lceil \sqrt{\psi(u)\psi(v)} \rceil$	$\{k+3, k+7, \dots, k+4n-5\}$

Then the set of edge labels is  $\{k, k+1, \dots, k+4n-5 = k+q-1\}$ . Hence  $Q_n$  is a  $k$ -geometric mean graph.  $\square$

**Example 8.** A 6-geometric mean labeling of  $Q_4$  is given below.



## 2.3 Join of graphs

From Theorem 17, we certain that a graph with more than one component from the set of stars, paths, combs, triangle snakes, or quadrilateral snakes

cannot be a geometric mean graph. Here we state a more general result on finite join of graphs.

**Theorem 25.** *A graph  $G$  obtained from a finite join of cycles and crowns, join with at most one path, comb, triangle snake, or quadrilateral snake, is a geometric mean graph.*

*Proof.* Let  $G$  be a graph with  $n$  components, obtained from a finite join of cycles and crowns, join with at most one path, comb, triangle snake, or quadrilateral snake.

Let  $G_1, G_2, \dots, G_n$  be  $n$  pairwise distinct components of  $G$ , i.e.  $G = G_1 + G_2 + \dots + G_n$ , and let  $q_i = |E(G_i)|$ , for  $1 \leq i \leq n$ . If there exists a component of  $G$  that is a path, a comb, a triangle snake, or a quadrilateral snake, let that component be  $G_n$ .

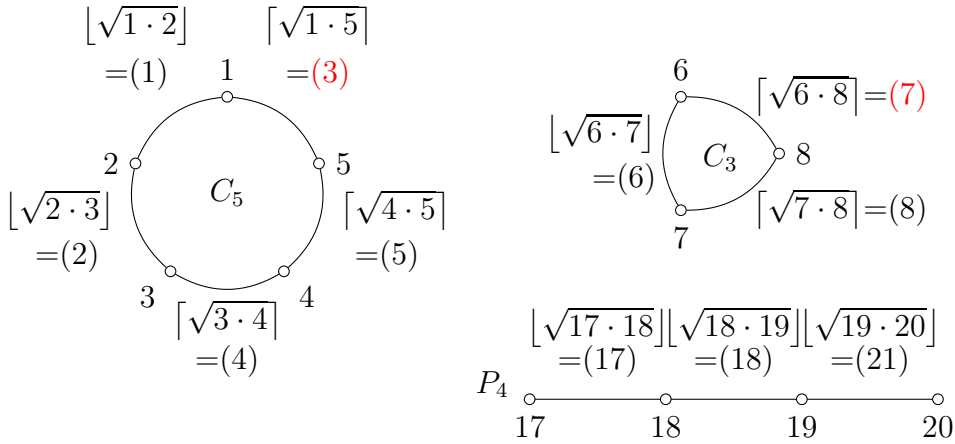
Let  $q_0 = 0$ . From Theorems 19, 20, 22, 21, 23 and 24,  
 $G_i$  is a  $\left(\sum_{j=0}^{i-1} q_j + 1\right)$ -geometric mean graph for any  $j \in \{1, 2, \dots, n\}$ .

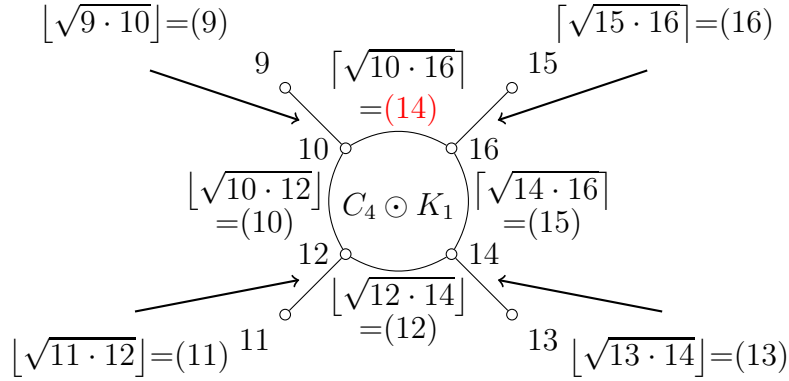
Moreover, with the  $\left(\sum_{j=0}^{i-1} q_j + 1\right)$ -geometric mean labeling constructed as in the proof of Theorems 20 and 21 the vertices labels set of  $G_i$  are disjoint and hence the result follows.  $\square$

By setting  $q_0$  in the proof of the above Theorem to  $k - 1$ , we obtain the following corollary.

**Corollary 26.** *A graph  $G$  obtained from a finite join of cycles and crowns, join with at most one path, comb, triangle snake, or quadrilateral snake, is a  $k$ -geometric mean graph.*

**Example 9.** A geometric mean labeling of  $C_5 + C_3 + (C_4 \odot K_1) + P_4$  is given below.





### 3 Conclusion

A new type of labeling, namely  $k$ -geometric mean labeling, is introduced in this paper as a tool to prove the geometric mean labeling of joins of graphs. We first presented some new geometric mean graph. Then, we define  $k$ -geometric mean labeling and study the geometric mean labeling on some classes of graphs. Finally, we provided a general result on finite joins of graphs.

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